Masters and Ph.D. Qualifying Exam **Analysis: Math 825/826**

Wednesday, Jan 22, 2020

• Work 5 of 6 problems. • Each problem is worth 20 points. • Use one side of the paper only and hand your work in order.

• Do not interpret a problem in such a way that it becomes trivial.

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(1) Let $\{a_n\}_{n=1}^{\infty} \subset (0, \infty)$ and $c > 0$ be given. Suppose that $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n$ diverges. Prove that there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} a_{n_k} = c$.

(2) Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be bounded sequences, and define the sets

$$A := \{a_n : n = 1, 2, \dots\}, \quad B := \{b_n : n = 1, 2, \dots\}, \quad \text{and} \quad C := \{a_n + b_n : n = 1, 2, \dots\}.$$

Prove or provide a counterexample each of the following statements.

(a) If $a \in \mathbb{R}$ is a limit point for A and $b \in \mathbb{R}$ is a limit point for B, then $a + b$ is a limit point for C. (Here limit point means accumulation or cluster point.)

(b) If $c \in \overline{C}$, then there exists $a \in \overline{A}$ and $b \in \overline{B}$ such that $a + b = c$.

(c) If $a_n \geq 0$ for all $n = 1, 2, \dots$, then $\limsup_{n\to\infty} (a_n^2) = (\limsup_{n\to\infty} a_n)^2$.

(3) Let (X, ρ) be a metric space and define $\sigma : X \times X \to [0, \infty)$ by

$$\sigma(x, y) := \min\{1, \rho(x, y)\}.$$

(a) Prove that σ is a metric on X.

(b) Prove that (X, ρ) is complete if and only if (X, σ) is complete.

(4) With $a < b$, let $\mathcal{C}([a, b])$ denote the family of all $\mathbb{R}$-valued functions that are continuous on the interval $[a, b]$.

(a) Let $M < \infty$ and $\mathcal{F} \subseteq \mathcal{C}([a, b])$ be given. Assume that each $f \in \mathcal{F}$ is differentiable on (a, b) and satisfies $|f(a)| \leq M$ and $|f'(x)| \leq M$ for all $x \in (a, b)$. Prove that $\mathcal{F}$ is equicontinuous on $[a, b]$.

(b) Let a uniformly bounded sequence of functions $\{g_n\}_{n=1}^{\infty} \subset \mathcal{C}([0, 1])$ be given. For each $n = 1, 2, \dots$, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} 0, & x = 0 \\ \dfrac{1}{x} \displaystyle\int_0^x s g_n(s) \mathrm{d}s, & 0 < x \leq 1. \end{cases}$$

Prove that there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ that converges uniformly on $[0, 1]$ to some $f \in \mathcal{C}([0, 1])$.

(5) (a) Let $a, b \in \mathbb{R}$, with $a < b$, be given, and suppose that $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and that $\lim_{x\to c} f'(x)$ both exists and is finite, for all $c \in (a, b)$. Prove that f is continuously differentiable on (a, b).

(b) Produce a function $f : (-1, 1) \to \mathbb{R}$ that is everywhere differentiable and such that f' is discontinuous at some $c \in (-1, 1)$. Justify your claim.

(6) The parts of this problem are not connected.

(a) Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and a strictly increasing sequence $\{x_n\}_{n=1}^{\infty} \subset (0, 1)$ be given. Assume that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and define $\alpha : [0, 1] \to \mathbb{R}$ by

$$\alpha(x) := \begin{cases} a_n, & x = x_n, \\ 0, & \text{otherwise}. \end{cases}$$

Prove or disprove: α has bounded variation on $[0, 1]$.

(b) Suppose that $f : [0, 1] \to \mathbb{R}$ is Riemann-Stieltjes integrable with respect to a nondecreasing function $\beta : [0, 1] \to [0, \infty)$. Prove that f is Riemann-Stieltjes integrable with respect to the function β^2.